7. SOLVING POLYNOMIAL EQUATIONS

§7.1. Early Solutions to Polynomial Equations

This chapter is a side-track from Galois Theory and, if you want to skip it, it won't do you any harm in terms of understanding the subsequent chapters. Never mind what Galois, himself, might say, Galois Theory is really all about fields, not polynomials. The insolubility of the quintic is just one minor application of Galois Theory, but it makes a nice end-point for our course.

However, since we've claimed that polynomials up to quartics can be solved by radicals, it would be nice to go some way towards finding out how to solve them. Besides, it's all very well to say that we can solve any real polynomial using Newton's Method. Sure, we can find the *real* zeros that way, to any desired degree of accuracy, but what about non-real zeros?

The Babylonians were able to solve what we would, today, call a quadratic equation, though the method was expressed in words, rather than the neat formula that we'd recognise. Algebraic equations weren't formulated symbolically until the 15th century.

And, of course, until negative numbers and imaginary numbers were introduced, non-real and even negative solutions were ignored because, such 'impossibilities' didn't exist. The lack of negative numbers even meant the need for several variations of the quadratic formula, because it was considered that there are three types of quadratic equation: $x^2 = bx + c$, $x^2 + c = bx$ and $x^2 + bx = c$. Here b, c are, of course, positive.

The cubic equation formula and the quartic formula both appeared in the 16th century. We'll develop the quadratic and cubic formula using the ideas of symmetry, that are important in Galois Theory. But feel free to read this chapter lightly, without getting bogged down in details, or even skipping it altogether. It contains one or two little ideas but by and large it's 'hard slog' algebraic manipulation.

§7.2. The Quadratic From An Advanced Standpoint

The solutions to $ax^2 + bx + c = 0$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The usual method of obtaining this formula involves a method called *completing the square* which doesn't generalize to higher degree polynomials. The following derivation of the quadratic formula gets more to the heart of the matter.

Let the zeros of $ax^2 + bx + c = 0$ be α and β. Then it's well known that the sum of the zeros and the product of the zeros can be expressed very simply in terms of the coefficients:

$$S = \alpha + \beta = -\frac{b}{a}, \quad P = \alpha\beta = \frac{c}{a}$$

Both the sum of the zeros and the product of the zeros are symmetric in terms of α and β. If α and β are swapped they remain unchanged.

These two functions of the zeros are called the **elementary symmetric functions** and other symmetric functions of the zeros can be expressed in terms of them.

Example 1: Express each of the following symmetric functions in terms of the elementary symmetric ones:

(a) $\alpha^2\beta + \beta^2\alpha$; (b) $\frac{1}{\alpha} + \frac{1}{\beta}$; (c) $\alpha^2 + \beta^2$;

(d) $\alpha^3 + \beta^3$.

Solution:

(a) $\alpha^2\beta + \beta^2\alpha = \alpha\beta(\alpha + \beta) = PS$;

(b) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{S}{P}$;

(c) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = S^2 - 2P$;

(d) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3(\alpha^2\beta + \alpha\beta^2) = S^3 - 3PS$.

It's obvious that any function which can be expressed in terms of $\alpha + \beta$ and $\alpha\beta$ must be symmetric in α and β. What is a little less well known is the fact that the converse holds. Every symmetric function in α and β can be expressed in terms of $\alpha + \beta$ and $\alpha\beta$. It follows that the value of such functions can be computed directly from the coefficients without having to solve the quadratic. (We don't provide a proof of this here.)

Now an expression such as $\alpha - \beta$ is not symmetric. Swapping α and β in fact changes the sign of the expression. However if we square $\alpha - \beta$, this change of sign disappears and we again get the symmetric function:

$$(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta = (\alpha + \beta)^2 - 4\alpha\beta = S^2 - 4PS$$

$$= \left(-\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right) \quad = \frac{b^2 - 4ac}{a^2}.$$

Hence we can find the values of $\alpha - \beta$ simply by taking square roots, getting $\alpha - \beta = \dfrac{\pm\sqrt{b^2 - 4ac}}{a}$.

Now $\alpha + \beta = -\dfrac{b}{a}$ and so adding these equations and dividing by 2 we get $\alpha = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

This formula can be expressed as an algorithm:

(1) Find $\Delta = b^2 - 4ac$;
(2) Solve $z^2 = \Delta$;
(3) Solve $2ax = z - b$.

Each step involves solving a linear equation or finding an n'th root.

§7.3. The Cubic Equation

The general cubic equation has the form

$$ax^3 + bx^2 + cx + d = 0$$

where $a \neq 0$. Let the zeros be α, β and γ. Then the elementary symmetric functions of these zeros can be expressed directly in terms of the coefficients as follows:

$$\mathbf{S = \alpha + \beta + \gamma = -\frac{\mathit{b}}{\mathit{a}},}$$

$$\mathbf{Q = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{\mathit{c}}{\mathit{a}},}$$

$$\mathbf{P = \alpha\beta\gamma = -\frac{\mathit{d}}{\mathit{a}}.}$$

As before, any symmetric function can be expressed in terms of these. For example:

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$
$$= S^2 - 2Q.$$

Example 2: Express the following symmetric functions in terms of P, Q, S:

(a) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$; (b) $\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha + \alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2$;

(c) $\alpha^3 + \beta^3 + \gamma^3$; (d) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$.

Solution:

(a) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma} = \frac{Q}{P}$;

(b) $\alpha^2\beta + \alpha\beta^2 + \alpha^2\gamma + \alpha\gamma^2 + \beta^2\gamma + \beta\gamma^2$

$= (\alpha\beta + \alpha\gamma + \beta\gamma)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma = QS - 3P.$

(c) $\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3$
$\quad - 3(\alpha^2\beta + \alpha\beta^2 + \alpha^2\gamma + \alpha\gamma^2 + \beta^2\gamma + \beta\gamma^2) - 6\alpha\beta\gamma$
$= S^3 - 3(QS - 3P) - 6P$
$= S^3 - 3QS + 3P.$

Other expressions have partial symmetry. For example consider:

$$\boldsymbol{\Delta_1 = \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha} \text{ and}$$
$$\boldsymbol{\Delta_2 = \alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2.}$$

These aren't completely symmetric because under the permutation $(\alpha\ \beta)$ the expressions Δ_1 and Δ_2 change into one another. But Δ_1 and Δ_2 are symmetric under the permutations $(\alpha\ \beta\ \gamma)$ and its inverse $(\alpha\ \gamma\ \beta)$. Including the identity permutation which keeps all of α, β and γ fixed we find that Δ_1 and Δ_2 are unchanged by three of the 6 permutations but are swapped by the other three. We could say that they have *half*-symmetry.

There are other expressions that have this half-symmetry. One notable example is the **discriminant**:

$$\boldsymbol{\Delta = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)}$$

which can be written as $\Delta_2 - \Delta_1$. For three of the 6 permutations on $\{\alpha, \beta, \gamma\}$ Δ is left fixed and for the other three Δ is sent to $-\Delta$.

This is just what we had with the quadratic discriminant. If we now *square* the discriminant we get something that is fully symmetric. And being fully symmetric we can express Δ^2 in terms of the elementary symmetric functions S, Q and P and hence we can find Δ^2

in terms of the coefficients. All we have to do is to take the square root and we've found Δ.

$\Delta_1 = \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha$;

$\Delta_2 = \alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2$;

$$\Delta_1 + \Delta_2 = (\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma$$
$$= QS - 3P ;$$

$$\Delta_1\Delta_2 = (\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha)(\alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2)$$
$$= (\alpha^3\beta^3 + \alpha^2\beta^2\gamma^2 + \alpha^4\beta\gamma) + (\alpha\beta^4\gamma + \beta^3\gamma^3 + \alpha^2\beta^2\gamma^2) + (\alpha^2\beta^2\gamma^2 + \alpha\beta\gamma^4 + \alpha^3\gamma^3)$$
$$= \alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 + 3P^2 + P(\alpha^3 + \beta^3 + \gamma^3)$$
$$= \alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 + 3P^2 + P(S^3 - 3QS + 3P).$$

Let S_1, Q_1, P_1 be the respective elementary symmetric functions in $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$.

Then $S_1 = Q$,

$$Q_1 = (\alpha\beta)(\beta\gamma) + (\beta\gamma)(\gamma\alpha) + (\gamma\alpha)(\alpha\beta) = PS,$$
$$P_1 = P^2.$$

Since $\alpha^3 + \beta^3 + \gamma^3 = S^3 - 3QS + 3P$

$$\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 = S_1^3 - 3Q_1S_1 + 3P_1$$
$$= Q^3 - 3SPQ + 3P^2.$$

Hence $\Delta_1\Delta_2$

$$= Q^3 - 3SPQ + 3P^2 + 3P^2 + PS^3 - 3SPQ + 3P^2$$
$$= PS^3 + Q^3 - 6SPQ + 9P^2.$$

$$(\Delta_1 - \Delta_2)^2 = (\Delta_1 + \Delta_2)^2 - 4\Delta_1\Delta_2$$
$$= (QS - 3P)^2 - 4(PS^3 + Q^3 - 6SPQ + 9P^2)$$

$$= Q^2S^2 - 27P^2 - 4PS^3 + 18SPQ - 4Q^3.$$

From these equations we can find Δ_1 and Δ_2.

Example 3: Find Δ_1 and Δ_2 for the polynomial

$$x^3 - 3x - 2.$$

Solution: $S = 0$, $Q = -3$, $P = 2$.

$\Delta_1 + \Delta_2 = QS - 3P = -6$;

$(\Delta_1 - \Delta_2)^2 = Q^2S^2 - 27P^2 - 4PS^3 + 18SPQ - 4Q^3$

$= -108 + 108 = 0$

Hence $\Delta_1 - \Delta_2 = 0$ and so $\Delta_1 = \Delta_2 = -3$.

For the quadratic equation, the role of Δ_1 and Δ_2 was played by the zeros α and β themselves. But with the cubic, we have a bit more work to do.

Let $E = \alpha + \beta\omega + \gamma\omega^2$

and $F = \alpha + \beta\omega^2 + \gamma\omega$.

These aren't even half-symmetric because the 3-cycle $(\alpha\ \beta\ \gamma)$ changes E to ω^2E and changes F to $F\omega$. But if we cube E and F then the ω and ω^2 will disappear. This means that E^3 and F^3 are half-symmetric. Maybe we can express them in terms of S, Q, P plus the half-symmetric expressions Δ_1 and Δ_2. If so then we can take cube roots to find the values of E and F.

Now in fact: $E^3 = (\alpha + \beta\omega + \gamma\omega^2)^3$

$= S^3 - 3QS + 9P + 3\omega^2\Delta_1 + 3\omega\Delta_2$ and

$F^3 = (\alpha + \beta\omega^2 + \gamma\omega)^3$

$= S^3 - 3QS + 9P + 3\omega\Delta_1 + 3\omega^2\Delta_2.$

Example 4: Find E and F for the polynomial

$$x^3 - 3x - 2.$$

Solution: In example 3 we found that:

$S = 0$, $Q = -3$, $P = 2$ and $\Delta_1 = \Delta_2 = -3$.

$$E^3 = S^3 - 3QS + 9P + 3\omega^2\Delta_1 + 3\omega\Delta_2$$
$$= 0 - 0 + 18 - 9\omega^2 - 9\omega$$
$$= 27 - 9(1 + \omega + \omega^2)$$
$$= 27$$

and similarly $F^3 = 27$.

Thus we have three possibilities for each of E and F:

$$E, F = 3, 3\omega \text{ or } 3\omega^2.$$

There are 9 combinations of these values but not all of them will produce solutions because

$$EF = (\alpha + \beta\omega + \gamma\omega^2)(\alpha + \beta\omega^2 + \gamma\omega)$$
$$= \alpha^2 + \beta^2 + \gamma^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)(\omega + \omega^2)$$
$$= S^2 - 2Q - Q$$
$$= S^2 - 3Q.$$

So provided $E \neq 0$ we can find F as $\dfrac{S^2 - 3Q}{E}$.

If $E = 0$ we must find F^3 and take cube roots.

So in fact we only have 3 possibilities for E, F. It is easy to show that any one will do. The other cases simply give the zeros in different orders.

Now having values for E and F, how do we get our hands on the zeros α, β and γ themselves? That's easy!

We have:

$$\begin{aligned}\alpha + \beta \quad + \gamma \quad &= S\\ \alpha + \beta\omega + \gamma\omega^2 &= E\ ;\\ \alpha + \beta\omega^2 + \gamma\omega &= F.\end{aligned}$$

If we simply add these equations, and use the relationship $1 + \omega + \omega^2 = 0$, we get $3\alpha = S + E + F$ and so

$$\alpha = \frac{S + E + F}{3}\ .$$

Similarly $\beta = \dfrac{S + E\omega^2 + F\omega}{3}$ and $\gamma = \dfrac{S + E\omega + F\omega^2}{3}$.

Example 5: Solve the cubic $x^3 - 3x - 2 = 0$.

Solution: Remember that $S = 0$, so $\alpha = \dfrac{E + F}{3}$.

Taking $E = F = 3$ we get $\alpha = 2$ and $\beta = \gamma = -1$.

We can describe the process of solving a cubic in the following table. The second and third columns give two examples.

CUBIC EQUATION

GENERAL CASE	EXAMPLE
$ax^3 + bx^2 + cx + d$	$x^3 - 4x^2 + 4x - 3$
$S = -b/a$	4
$Q = c/a$	4
$P = -d/a$	3
$\Delta_1 + \Delta_2 = QS - 3P$	7
$(\Delta_1 - \Delta_2)^2 = Q^2S^2 - 27P^2 - 4PS^3 + 18SPQ - 4Q^3$	–147
$\Delta_1 - \Delta_2$ = either square root of the above	$7\sqrt{3}i$
$\Delta_1 = \dfrac{(\Delta_1 + \Delta_2) + (\Delta_1 - \Delta_2)}{2}$	$\dfrac{7(1 + \sqrt{3}i)}{2} = -7\omega^2$
$\Delta_2 = (\Delta_1 + \Delta_2) - \Delta_1$	$\dfrac{7(1 - \sqrt{3}i)}{2} = -7\omega$
$E^3 = S^3 - 3QS + 9P + 3\omega^2\Delta_1 + 3\omega\Delta_2$	64
$F^3 = S^3 - 3QS + 9P + 3\omega\Delta_1 + 3\omega^2\Delta_2$	1
$EF = S^2 - 3Q$	4
E = any cube root of E^3	4
F = (EF)/E or any cube root of F^3	1
$\alpha = \dfrac{S + E + F}{3}$	3
$\beta = \dfrac{S + E\omega + F\omega^2}{3}$	$1 + \omega$
$\gamma = S - (\alpha + \beta)$	$-\omega$

NOTE: The cube roots must match so that

$$EF = S^2 - 3Q.$$

If $E \neq 0$ calculate F as (EF)/E. (In this case there is no need to calculate F^3.)

However if $E = 0$ then F must be calculated as a cube root of F^3.

GENERAL CASE	EXAMPLE
$ax^3 + bx^2 + cx + d$	$x^3 - 6x^2 + 12x + 9$
$S = -b/a$	6
$Q = c/a$	–12
$P = -d/a$	9
$\Delta_1 + \Delta_2 = QS - 3P$	45
$(\Delta_1 - \Delta_2)^2 = Q^2S^2 - 27P^2 - 4PS^3 + 18SPQ - 4Q^3$	–27
$\Delta_1 - \Delta_2$ = either square root of the above	$3\sqrt{3}i$
$\Delta_1 = \dfrac{(\Delta_1 + \Delta_2) + (\Delta_1 - \Delta_2)}{2}$	$\dfrac{45 + 3\sqrt{3}i}{2}$
$\Delta_2 = (\Delta_1 + \Delta_2) - \Delta_1$	$\dfrac{45 - 3\sqrt{3}i}{2}$
$E^3 = S^3 - 3QS + 9P + 3\omega^2\Delta_1 + 3\omega\Delta_2$	0
$F^3 = S^3 - 3QS + 9P + 3\omega\Delta_1 + 3\omega^2\Delta_2$	–27
$EF = S^2 - 3Q$	0
E = any cube root of E^3	0
F = (EF)/E or any cube root of F^3	–3

$\alpha = \dfrac{S + E + F}{3}$	3
$\beta = \dfrac{S + E\omega + F\omega^2}{3}$	$2 + \omega^2$
$\gamma = S - (\alpha + \beta)$	$2 + \omega$

It's possible to summarize the whole process into a single formula as follows. Firstly, in order to keep the formula simple, we divide through by the coefficient of x^3 to get the cubic in the form $x^3 - Sx^2 + Qx - P$, so that S is the sum of the zeros etc. Then, observing that the transformation $y = x - S/3$ leads to a cubic with no x^2 term, we can (without loss of generality) consider cubic equations of the form $x^3 + Qx - P = 0$

The zeros can be expressed by the formula:

$$\boldsymbol{\alpha, \beta, \gamma = \sqrt[3]{\frac{P}{2} + \sqrt{\frac{P^2}{4} + \frac{Q^3}{27}}} + \sqrt[3]{\frac{P}{2} - \sqrt{\frac{P^2}{4} + \frac{Q^3}{27}}}}$$

The cube roots are computed over the complex field and are chosen so that the product of these is – Q/3.

Example 6: Solve $x^3 - 6x - 6 = 0$.
Solution: P = 6, Q = –6 so one solution is $\sqrt[3]{4} + \sqrt[3]{2}$ (This is the only real solution.)

While it's quite neat to have a single formula, it's a little easier to use the algorithm in practice. However we presented this account of the solution of the cubic not because you'll be solving cubics in real life! If you had to do that you'd probably use Newton's Method.

Example 7: Solve the cubic $x^3 + x + 1 = 0$.
Solution: Using Newton's Method there's a real zero at

$$x = -0.682378...$$

Factorising $x^3 + x + 1$ by $x + 0.682378$ we get (approximately) $x^2 - 0.682378x + 1.465571$.
Solving this quadratic we get the remaining two zeros as 0.341189 ± 1.161534 i.

Now feel free to jump over to chapter 8 at this point. The rest of this chapter has nothing to do with Galois Theory whatsoever. All it does is to provide techniques for solving quartics and quintics, or at least getting good approximations to their solutions, using Newton's Method, especially in the case where there are two pairs of conjugate zeros.

§7.4. Quartic Equations

There's a formula for the quartic that can be derived along the same lines as for the quadratic and cubic, but we won't present it here. We would never need exact solutions and so we use Newton's Method.

However Newton's Method only finds approximations to real zeros. This presents a problem when there are non-real zeros.

We're assuming that our quartic has real coefficients and so we need to be able to deal with the case where there are two pairs of conjugate zeros.

To simplify things we'll assume that we have a monic quartic with no x^3 term. For any monic quartic $x^4 + ax^3 + bx^2 + dx + e$, substituting $y = x - a/4$ will give such a quartic.

Example 8: Make a suitable substitution $y = x - k$ to transform $x^4 + 8x^3 + 3x + 7$ into the form

$$y^4 + ey^2 + fy + g.$$

Solution: Letting $y = x - k$ transforms the quartic to

$(y + k)^4 + 8(y + k)^3 + 3(y + k) + 7$

$= y^4 + 4ky^3 + 6k^2y^2 + 4k^3y + k^4 + 8y^3 + 24ky^2 + 24k^2y + 8k^3 + 3y + 3k + 7$

$= y^4 + 4(k + 2)y^3 + 6k(k + 3)y^2 + (4k^3 + 24k^2 + 3)y + (k^4 + 8k^3 + 7).$

Let $k = -2$.

Then the polynomial becomes $y^4 - 12y^2 + 67y - 41$.

We now consider a quartic of the form:

$$x^4 + Qx^2 - Rx + P.$$

If the zeros are α, β, γ, δ then $\alpha + \beta + \gamma + \delta = 0$ and

$Q = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$,

$R = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$,

$P = \alpha\beta\gamma\delta$.

Theorem 1 (COOPER): Let $f(x) = x^4 + Qx^2 - Rx + P$ where Q, R, P are real and $R \neq 0$.
If $f(x)$ has 4 non-real zeros they are $a + b$i where

$$a = \pm \frac{\sqrt{X}}{2},\ b = \pm \sqrt{\frac{X + 2Q}{4} - \frac{R}{4a}}$$

and where X is a positive real zero of the cubic:

$$X^3 + 2QX^2 + (Q^2 - 4P)X - R^2.$$

Proof: Let the zeros be:

$\alpha = a + b\mathrm{i}, \quad \beta = a - b\mathrm{i}, \quad \gamma = c + d\mathrm{i}, \quad \delta = c - d\mathrm{i}.$

Then $2(a + c) = 0$ so $c = -a$.
$Q = a^2 + b^2 + c^2 + d^2 + 4ac = b^2 + d^2 - 2a^2$, since $c = -a$.
$R = 2a(d^2 - b^2)$
$P = (a^2 + b^2)(a^2 + d^2)$.
So we must solve the non-linear system:

$$\left.\begin{aligned} b^2 + d^2 - 2a^2 &= Q \\ 2a(d^2 - b^2) &= R \\ (a^2 + b^2)(a^2 + d^2) &= P \end{aligned}\right\}$$

From the middle equation we get $2ad^2 = 2ab^2 + R$.
Multiplying the first equation by $2a$ and substituting for $2ad^2$ we get $4ab^2 + R - 4a^3 = 2Qa$ and so
$4ab^2 = 4a^3 + 2Qa - R$.
Multiplying the third equation by $16a^2$ and substituting for $2ad^2$ we get:

$$(4a^3 + 4ab^2 - R)(4a^3 + 4ab^2 + R) = 16Pa^2.$$

Substituting for $4ab^2$ we get:

$$(8a^3 + 2Qa - R)(8a^3 + 2Qa + R) = 16Pa^2.$$

Hence $(8a^3 + 2Qa)^2 - R^2 = 16Pa^2$

and so $64a^6 + 32Qa^4 + 4(Q^2 - 4P)a^2 - R^2 = 0$.

Let $X = 4a^2$. Then $X^3 + 2QX^2 + (Q^2 - 4P)X - R^2 = 0$.
We solve this cubic for a positive real solution X, by Newton's Method.
Then $a = \pm \frac{\sqrt{X}}{2} \neq 0$ and $c = -a$.
Now $4ab^2 = 4a^3 + 2Qa - R = a(4a^2 + 2Q) - R$.
Since $a \neq 0$, $b = \pm\sqrt{\frac{X + 2Q}{4} - \frac{R}{4a}}$ and

$d^2 = \frac{X + 2Q}{4} + \frac{R}{4a}$ so $d = \pm\sqrt{\frac{X + 2Q}{4} + \frac{R}{4a}}$.

In the case where R = 0, the quartic is just a quadratic in x^2 so its zeros are easy to find.

Example 9: The quartic $z^4 + 4z^3 + 8z^2 + z + 1$ has no real zeros. Solve it over ℂ.
Solution: Let $x = z + 1$. Therefore

$$(x - 1)^4 + 4(x - 1)^3 + 8(x - 1)^2 + (x - 1) + 1 = 0$$

and so $x^4 - 4x^3 + 6x^2 - 4x + 1 + 4x^3 - 12x^2 + 12x - 4 +$
$8x^2 - 16x + 8 + x = 0$

Hence $x^4 + 2x^2 - 7x + 5 = 0$. So Q = 2, R = 7, P = 5.
We find a positive real zero of $X^3 + 4X^2 - 16X - 49 = 0$ by Newton's Method. This is $X = 3.750278$.
The zeros of $x^4 + 2x^2 - 7x + 5$ are of the form $a + b\text{i}$ where
$a = \pm \frac{\sqrt{X}}{2} = \pm 0.968282$.

If $a = 0.968282$ then

$$b = \pm\sqrt{\frac{X + 2Q}{4} - \frac{R}{4a}} = \pm\, 0.360895.$$

If $a = -\,0.968282$ then $b = \pm\, 1.935173$.
So the zeros of $x^4 + 2x^2 - 7x + 5$ are:
0.968282 ± 0.360895, and
$-\,0.968282 \pm 1.935173$.
Hence the zeros of $z^4 + 4z^3 + 8z^2 + z + 1$ are:
$-\,0.031718 \pm 0.360895$ and
$-\,1.968282 \pm 1.935173$.

§7.5. Solving Quintics

We will be showing that it is impossible to solve a general quintic – at least not if we want exact solutions derived from the coefficients by the operations of addition, subtraction, multiplication, division and extraction of roots.

But if we simply want an approximation, to any desired degree of accuracy, then it is indeed possible. Having odd degree there will be at least one real zero which we can find by Newton's Method. We can divide by the corresponding linear factor to get a quartic.

Example 10: Solve the quintic $f(x) = 3x^5 - 5x^3 + 3 = 0$.
Solution: Starting with $x_0 = -2$ we use Newton's Method to find a real zero: $x = -1.41986458$.
A graph reveals that this is the only real zero.
Dividing $f(x)$ by $x + 1.41986458$ we get

$3x^4 - 4.25959314x^3 + 1.048046277x^2 - 1.488083787x$
$+ 2.112877461 = 0.$

Hence $x^4 - 1.41986438x^3 + 0.349348759x^2$
$- 0.496027929x + 0.704292487 = 0.$

Let $y = x - \dfrac{1.41986438}{4} = x - 0.354966095.$

Substituting x = y + 0.354966095 we get:

So $y^4 + 0.733265399y^2 - 0.465002974y$
$+ 0.588050385 = 0.$

Let Q = 0.733265399,
R = 0.465002974,
P = 0.588050385.

We solve the cubic $X^3 + 2QX^2 + (Q^2 + 4P)X - R^2 = 0$. That becomes:

$X^3 + 1.466530798X^2 - 1.814523395X - 0.216227766$
$= 0.$

Using Newton's Method we get X = 0.878656046.

So $a = 0.468683274$, $c = -0.468683274$.
$b = 0.581600956$, $d = 0.913418716$.

So the zeros of $y^4 + 0.733265399y^2 - 0.465002974y$
$+ 0.588050385 = 0$ are:

$0.468683274 \pm 0.581600956$ i and
$-0.468683274 \pm 0.913418716$ i.

Since $x = y + 0.354966095$ the zeros of $3x^5 - 5x^3 + 3$ are (approximately):

$$-1.41986458,$$
$$0.823649369 \pm 0.581600956\,\text{i and}$$
$$-0.113717179 \pm 0.913418716\,\text{i}.$$

EXERCISES FOR CHAPTER 7

Exercise 1:

Suppose α, β are the zeros of a quadratic $ax^2 + bx + c$ and let $S = \alpha + \beta$, $P = \alpha\beta$.

(i) Show that $\alpha^5 + \beta^5 = S^5 - 5PS^3 + 5P^2S$.

(ii) Express $K = \dfrac{\alpha}{\alpha^3 + \beta^2} + \dfrac{\beta}{\beta^3 + \alpha^2}$ in terms of S and P.

Exercise 2:

Suppose α, β, γ are the zeros of a cubic and let $S = \alpha + \beta + \gamma$, $Q = \alpha\beta + \alpha\gamma + \beta\gamma$ and
$P = \alpha\beta\gamma$. Express each of the following in terms of S, Q and P.

(i) $G = \alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta$.

(ii) $H = \alpha^5\beta^3\gamma + \alpha^5\gamma^3\beta + \beta^5\alpha^3\gamma + \beta^5\gamma^3\alpha + \gamma^5\alpha^3\beta + \gamma^5\beta^3\alpha$.

Exercise 3:

Let G be the group of all permutations on $\{\alpha, \beta, \gamma, \delta\}$.

(i) Show that $E = \alpha\beta + \gamma\delta$ is fixed by a subgroup of G which is isomorphic to D_8.

(ii) Let $F = \alpha + \beta i - \gamma - \delta i$. Prove that F^4 is fixed by the permutation $(\alpha\ \beta\ \gamma\ \delta)$.

Exercise 4: Solve each of these cubics by computing S, Q, P, Δ_1, Δ_2, E and F.

(i) $x^3 - 4x^2 + 4x - 3 = 0$;

(ii) $x^3 - 6x^2 + 12x + 3 = 0$.

Exercise 5: Solve the quartic $x^4 + x^2 + x + 1 = 0$.

SOLUTIONS FOR CHAPTER 7

Exercise 1: (i) $\Sigma\alpha^5 = (\Sigma\alpha)^5 - 5.\Sigma\alpha^4\beta - 10.\Sigma\alpha^3\beta^2$

$$= S^5 - 5P.\Sigma\,\alpha^3 - 10P^2S$$

Now $\Sigma\alpha^3 = (\Sigma\alpha)^3 - 3.\Sigma\alpha^2\beta = S^3 - 3PS$.

Thus:

$\Sigma\alpha^5 = S^5 - 5P(S^3 - 3PS) - 10P^2S = S^5 - 5PS^3 + 5P^2S$.

(ii) $E = \dfrac{\Sigma\alpha^3\beta + \Sigma\alpha^3}{\alpha^3\beta^3 + \alpha^2\beta^2 + \Sigma\alpha^5}$

$$= \frac{P(S^2 - 2P) + S^3 - 3PS}{P^3 + P^2 + (S^5 - 5PS^3 + 5P^2S)}$$

$$= \frac{PS^2 - 2P^2 + S^3 - 3PS}{P^3 + P^2 + S^5 - 5PS^3 + 5P^2S}$$

Exercise 2: (i) $G = \Sigma\alpha^2\beta = \Sigma\alpha\beta\,.\,\Sigma\alpha - 3P = QS - 3P$

(ii) $F = P\Sigma\alpha^4\beta^2$

$$= P[(\Sigma\alpha^2\beta)^2 - 2\Sigma\alpha^4\beta\gamma - 2\Sigma\alpha^3\beta^3 - 2\Sigma\alpha^3\beta^2\gamma - 6P^2]$$

$$= PG^2 - 2P^2\Sigma\alpha^3 - 2P\Sigma\alpha^3\beta^3 - 2P^2G - 6P^3$$

Now $\Sigma\alpha^3 = (\Sigma\alpha)^3 - 3\Sigma\alpha^2\beta - 6P = S^3 - 3G - 6P$ and

$\Sigma\alpha^3\beta^3 = (\Sigma\alpha\beta)^3 - 3\Sigma\alpha^3\beta^2\gamma - 6P^2 = Q^3 - 3PG - 6P^2$.

Thus $H = PG^2 - 2P^2(S^3 - 3G - 6P)$

$$- 2P(Q^3 - 3PG - 6P^2) - 2P^2G - 6P^3$$

$$= PG^2 - 2P^2S^3 + 6P^2G + 12P^3 - 2PQ^3 + 6P^2G$$

$$+ 12P^3 - 2P^2G - 6P^3$$

$$= PG^2 - 2P^2S^3 + 10P^2G + 18P^3 - 2PQ^3$$

$= 18P^3 - 2PQ^3 - 2P^2S^3 + 10P^2(QS - 3P) + P(QS - 3P)^2$
$= 18P^3 - 2PQ^3 - 2P^2S^3 + 10P^2QS - 30P^3$
$+ P(Q^2S^2 + 9P^2 - 6PQS)$
$= -3P^3 - 2PQ^3 - 2P^2S^3 + 4P^2QS + PQ^2S^2.$

Exercise 3: (i) E is fixed by $\{I, (\alpha\beta), (\gamma\delta), (\alpha\gamma\beta\delta), (\alpha\beta)(\gamma\delta), (\alpha\delta\beta\gamma), (\alpha\gamma)(\beta\delta), (\alpha\delta)(\beta\gamma)\}$. If $A = (\alpha\gamma\beta\delta)$ and $B = (\alpha\beta)$ then $G = \langle A, B \mid A^4 = B^2 = 1, BA = A^{-1}B\rangle$
(ii) $F = \alpha + \beta i - \gamma - \delta i \to \beta + \gamma i - \delta - \alpha i$
$= -i(\alpha + \beta i - \gamma - \delta i)$
$= -iF$ so $F^4 \to F^4$.

Exercise 4:
(i) $S = Q = 4$, $P = 3$,
$\Delta_1 + \Delta_2 = SQ - 3P = 7$,\
$(\Delta_1 - \Delta_2)^2 = S^2 - 27P^2 + 18SPQ - 4Q^3 - 4PS^3.$
$= -147$
so let $\Delta_1 - \Delta_2 = \sqrt{147}\, i = 7\sqrt{3}\, i$.
Hence $\Delta_1 = \dfrac{7 + 7\sqrt{3}i}{2} = -7\omega^2$ and $\Delta_2 = -7\omega$.
$E^3 = S^3 - 3QS + 9P + 3\omega^2\Delta_1 + 3\omega\Delta_2$
$= 43 - 21\omega - 21\omega^2$
$= 43 + 21 = 64$ so take $E = 4$.
$EF = S^2 - 3Q = 4$ so $F = 1$.
The zeros are thus $\dfrac{4 + 4 + 1}{3} = 3$,
$\dfrac{4 + 4\omega + \omega^2}{3} = 1 + \omega = -\omega^2$ and

$$\frac{4 + 4\omega^2 + \omega}{3} = 1 + \omega^2 = \mathbf{-\,\omega}.$$

(ii) $S = 6$, $Q = 12$, $P = 9$. $\Delta_1 + \Delta_2 = 45$, $(\Delta_1 - \Delta_2)^2 = -\,27$ so take $\Delta_1 - \Delta_2 = 3\sqrt{3}\,i$.

Hence $\Delta_1 = \dfrac{45 + 3\sqrt{3}i}{2}$ and $\Delta_2 = \dfrac{45 - 3\sqrt{3}i}{2}$.

$E^3 = 0$ so $E = 0$. $F^3 = -\,3$ so take $F = -3$.

The zeros are thus 3, $2 + \omega$ and $2 + \omega^2$.

Exercise 5: $Q = 1$, $R = -1$, $P = 1$.

We solve $X^3 + 2QX^2 + (Q^2 - 4P)X - R^2$

$$= X^3 + 2X^2 - 3X - 1 = 0.$$

By Newton's Method $X = 1.198691244$.

Let $a = \dfrac{\sqrt{X}}{2} = 0.5474238$, $c = -\,a = -\,0.5474238$,

$$b = \sqrt{\frac{X + 2Q}{4} - \frac{R}{4a}} = 1.1208735,$$

$$d = \sqrt{\frac{X + 2Q}{4} + \frac{R}{4a}} = 0.5856512.$$

So the solutions are $0.5474238 \pm 1.1208735\,i$ and

$$-\,0.5474238 \pm 0.5856512\,i.$$